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**Results in Mathematics**

# Linear Graininess Time Scales and Ladder Operators of Orthogonal Polynomials

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**Abstract.** In this paper we introduce linear graininess (LG) time scales. We further study orthogonal polynomials (OPs) with the weight function supported on LG time scales and derive the raising and lowering ladder operators by using the time scales calculus. We also derive a second order dynamic equation satisfied by these polynomials. The notion of an LG time scale encompasses the cases of the reals, the $h$-equidistant grid, the $q$-grid and, more general, a mixed $(q, h)$-grid. This allows a unified treatment of the ladder operators theory for classical OPs on these time scales. Moreover we will explain, why exclusively LG time scales provide the right framework for general OP theory.

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**Keywords.** Time scales, linear graininess time scales, ladder operators, orthogonal polynomials.

## 1. Introduction

The time scales calculus unifies continuous and discrete analysis. Many analogous results in continuous and discrete analysis can be elegantly proved by using the techniques of time scales calculus. This provides an invaluable insight into many of the investigated problems. Similar proofs in distinct areas of differential and difference calculus can be simplified and be simultaneously written by using the time scales notation. In recent years there have been many studies on dynamic equations (which include differential and difference equations as particular cases) by using the time scales calculus.

The main objective of this paper is to introduce the linear graininess (LG) time scales and, as an application, show how the theory of ladder operators for orthogonal polynomials (OPs) can be studied by using the time scales calculus.

Orthogonal polynomials with the weight function supported on (a subset of) the real line have been extensively studied (see [1] and the references therein). They satisfy a three-term recurrence relation. The papers [2–5] focus on deriving the so-called raising and lowering operators (which raise and lower the degrees of the polynomials) for different types of OPs. The techniques used in all these cases are similar, they mainly rely on the use of the orthonormality condition, the three-term recurrence relation, the Christoffel–Darboux formula and integration by parts (or its analogue in the discrete setting). The raising and lowering operators involve two functions $A_n(x)$ and $B_n(x)$ which satisfy certain recurrence relations. Moreover, by knowing the raising and lowering operators one can easily derive the second order (differential, difference or $q$-difference) equation satisfied by the OPs. The second main objective of the current paper is to revisit the proofs [2–5] and to re-derive the ladder operators in a unified way by using the time scales calculus. Other articles [6,7] with the same flavor do not fit directly into the framework of time scales, since they are concerned with $q$-difference operators, whereas the underlying time scale is the set of real numbers $\mathbb{R}$, but not, as would be appropriate from the time scale point of view, the $q$-grid. In addition, our proof will cover the most general case of a $(q, h)$-grid (iterates of $t \mapsto qt + h$) that we will explain more accurately as time scales of Type III or IV in Theorem 1 below. We remark that the framework of time scales calculus was used in [8] to study the Hermite polynomials, functions and their generalizations on two different time scale structures with certain ladder formalism presented in both cases. In [9] the authors utilize time scales calculus to study the Laplace transform on the $h$-grid and $q$-grid simultaneously.

The time scales calculus provides a unification between the discrete and continuous analytic structures and has many applications in different areas of mathematical analysis. We also claim that the use of time scales formalism has a smoothing effect on quite technical and dodgy calculations. This is mainly due to the fact that almost all manipulations of formulas are performed on the "diff" level (differentiation, difference operator) rather than the "int" level (integration, sums).

**The Outline of the Paper.** We first survey the time scales calculus following [10–17]. Next we introduce the LG (linear graininess) time scales which we consider in this paper and introduce OPs on the time scales. Finally we prove the main results on ladder operators, which are differential, difference or $q$-difference operators in the continuous, discrete and quantum case, respectively. We also discuss how the results of this paper generalize the results of [2–5].

## 2. Time Scales: A Brief Survey

Commonly *a time scale* $\mathbb{T}$ is defined as a closed subset of the real line $\mathbb{R}$. If there is a minimal or maximal endpoint of $\mathbb{T}$ it can be removed without loss of generality. So we shall adopt the viewpoint that a time scale has the form $\mathbb{T} = I \backslash O$, where $I$ is an interval and $O$ is an open subset of $\mathbb{R}$.

The time scale can be equipped with its relative topology. For $r, s \in \mathbb{T}$ one can define the intervals $[r, s] := \{t \in \mathbb{T} | r \leq t \leq s\}$, and similarly one can define open, half open and unbounded intervals.

The *forward and backward jump operators* are mappings $\mathbb{T} \to \mathbb{T}$ defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \tag{1}$$

$$\varrho(t) := \sup\{s \in \mathbb{T} : s < t\} \tag{2}$$

respectively. This includes $t$ being maximal or minimal in $\mathbb{T}$, since $\inf \varnothing = -\infty$ and $\sup \varnothing = +\infty$.

A point $t \in \mathbb{T}$ is called right-dense, right-scattered, left-dense or left-scattered, if $\sigma(t) = t$, $\sigma(t) > t$, $\varrho(t) = t$ or $\varrho(t) < t$, respectively. A minimal/maximal point is also left/right-dense.

A simple but thorough consideration shows the equivalence of the following statements:

(A) $\mathbb{T}$ does not contain points that are left-dense and right-scattered.
(B) On $\mathbb{T}$ we have $\sigma \circ \varrho = \mathrm{id}_{\mathbb{T}}$.
(C) $\varrho$ is a one-to-one (injective) mapping.
(D) There is a left-inverse mapping for $\varrho$.
(E) $\sigma$ is an onto (surjective) mapping.
(F) There is a right-inverse mapping for $\sigma$.

Note that $\mathbb{T} = [0, 1] \cup \mathbb{N}$ (at $t = 1$) is a counterexample to each of the statements.

The *graininess function* $\mu^* : \mathbb{T} \to \mathbb{R}_0^+$ is defined as the gap length between a point and its right neighbor, i.e.,

$$\mu^*(t) := \sigma(t) - t. \tag{3}$$

A function $f : \mathbb{T} \to \mathbb{R}$ (or, more generally, with range in a Banach space) is called *differentiable at* $t$ with the derivative $f^\Delta(t)$, if for every $\varepsilon > 0$ there is a neighborhood $U$ of $t$ such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s| \quad \text{for all } s \in U. \tag{4}$$

Alternatively one can define

$$f^\Delta(t) := \lim_{s \to t,\, s \neq \sigma(t)} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}. \tag{5}$$

If a function $f$ is differentiable at $t$, then it is continuous at $t$. A quite simple but nevertheless useful formula is

$$f(\sigma(t)) = f(t) + \mu^*(t) \cdot f^\Delta(t). \tag{6}$$

The differentiation is linear and has the following product and quotient rules:

$$(f \cdot g)^\Delta(t) = f(\sigma(t)) \cdot g^\Delta(t) + f^\Delta(t) \cdot g(t), \tag{7}$$

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t) \cdot g(t) - f(t) \cdot g^\Delta(t)}{g(\sigma(t)) \cdot g(t)}. \tag{8}$$

Let $J \subseteq \mathbb{T}$ be an interval. A derivative of a differentiable function $f : J \to \mathbb{R}$ will have $J^\kappa$ as its domain, where the *kappa operator* (from German “Kappen”-truncation) cuts off a left-scattered (isolated) maximum, if present:

$$J^\kappa := J \backslash \{\text{left–scattered}\max J\}. \tag{9}$$

A function $f : J^\kappa \to \mathbb{R}$ is called *rd-continuous*, if it is continuous at every right-dense $t \in J^\kappa$ and has all one-sided limits at all points. Although we do not need either of the preceding two notions in this article, they are quite natural for integration of functions in the general time scales context.

For every rd-continuous function $f : J^\kappa \to \mathbb{R}$ there is an antiderivative, that is a differentiable function $F$ with

$$F^\Delta(t) \;=\; f(t) \quad \text{ for all } t \in J^\kappa.$$

It is unique up to an additive constant, since one can show that

$$F \equiv C \quad \Longleftrightarrow \quad F^\Delta \equiv 0.$$

If $\mathbb{T}$ is an infinite discrete time scale and the sum

$$F(t) \;=\; -\sum_{j=0}^{\infty} \mu^*(\sigma^j(t)) f(\sigma^j(t)) \tag{10}$$

is well-defined, then it is an antiderivative.

Next one can define the integral of the rd-continuous function $f : J^\kappa \to \mathbb{R}$ by

$$\int_r^s f(t)\Delta t = F(s) - F(r).$$

For $u \in J$ we have

$$\int_r^u f(t)\Delta t + \int_u^s f(t)\Delta t \;=\; \int_r^s f(t)\Delta t \;=\; -\int_s^r f(t)\Delta t.$$

Moreover, integration is linear and the integration by parts can be performed as follows:

$$\int_r^s f(\sigma(t)) g^\Delta(t)\,\Delta t + \int_r^s f^\Delta(t) g(t)\,\Delta t \;=\; f(s)g(s) - f(r)g(r). \tag{11}$$

## 3. Linear Graininess Time Scales

In this section we introduce LG (linear graininess) time scales, discuss their properties and write down corresponding integral expressions. At the end we briefly discuss the time reversal operator.

**Theorem 1.** *Let $q \in \mathbb{R}\backslash\{0\},\ h \in \mathbb{R}$. The following conditions for a time scale $\mathbb{T}$ are equivalent.*

(A) *The graininess is a linear function of the form $\mu^*(t) = (q-1)t+h$ on $\mathbb{T}$.*

(B) *The jump operators on $\mathbb{T}$ are affine mappings that are inverse of each other:*

$$\sigma(t) = qt + h \qquad \text{and} \qquad \varrho(t) = \sigma^{-1}(t) = \frac{t-h}{q}.$$

(C) *$\mathbb{T}$ is essentially one of the types* I–IV *of time scales $\mathbb{T}_{q,h,t_0}$ on the following list:*

| *Type* | $q$ | $h$ | $t_0$ | $\mathbb{T}_{q,h,t_0}$ | $\inf \mathbb{T}$ | $\sup \mathbb{T}$ |
|---|---|---|---|---|---|---|
| I | $1$ | $0$ | — | $\mathbb{R}$ | $-\infty$ | $+\infty$ |
| II | $1$ | $>0$ | $\in \mathbb{R}$ | $h\mathbb{Z} + t_0$ | $-\infty$ | $+\infty$ |
| III | $>1$ | | $> \frac{-h}{q-1}$ | $\{\sigma^j(t_0) \vert j \in \mathbb{Z}\}$ | $\frac{-h}{q-1}$ | $+\infty$ |
| IV | $<1$ | | $< \frac{h}{1-q}$ | $\{\sigma^j(t_0) \vert j \in \mathbb{Z}\}$ | $-\infty$ | $\frac{h}{1-q}$ |
| II’ | $1$ | $>0$ | $=0$ | $h\mathbb{Z}$ | $-\infty$ | $+\infty$ |
| III’ | $>1$ | $0$ | $=1$ | $q^{\mathbb{Z}}$ | $0$ | $+\infty$ |
| IV’ | $<1$ | $0$ | $=-1$ | $-q^{\mathbb{Z}}$ | $-\infty$ | $0$ |

*Here $\sigma^j = \varrho^{-j}$ is the $j$-fold application of the jump operator $\sigma(t) = qt+h$. “Essentially” means that one can modify a time scale $\mathbb{T}$ on this list by taking an interval $I \subseteq \mathbb{T}$ or by adding a finite* inf *or* sup*. Types III’, IV’ and V’ are well-known special cases of types III, IV, V, respectively.*

(D) *For each $n \in \mathbb{N}$ the derivative of a polynomial with degree $n$ is a polynomial with degree $n-1$.*

(E) *For each polynomial* $p$ *the* $\sigma$*-shift* $p \circ \sigma$ *and the* $\varrho$*-shift* $p \circ \varrho$ *are polynomials with the same degree.*

*Proof.* Implication (A) $\Longrightarrow$ (D). For $\mu \equiv 0$ we have $\mathbb{T} = \mathbb{R}$ and (D) is clear. So let $\mu^*(t) = (q-1)t + h > 0$. This implies that the time scale is discrete. Now we compute the derivative of a monomial $t^n$.

$$\begin{aligned}(t^n)^\Delta &= \frac{(\sigma(t))^n - t^n}{\mu^*(t)} = \frac{(t+\mu^*(t))^n - t^n}{\mu^*(t)} \\ &= \frac{1}{\mu^*(t)} \cdot \left[\sum_{j=0}^{n} \binom{n}{j} t^{n-j} \mu^*(t)^j - t^n\right] \\ &= \sum_{j=1}^{n} \binom{n}{j} t^{n-j} \mu^*(t)^{j-1} = n\ t^{n-1} + \mu^*(t) \sum_{j=2}^{n} \binom{n}{j} t^{n-j} \mu^*(t)^{j-2}.\end{aligned}$$

So we can see that $(t^n)^\Delta$ is a polynomial with degree $n-1$.

Implication (D) $\Longrightarrow$ (A). By (D) the derivative of $t^2$

$$(t^2)^\Delta = \lim_{s \to t,\, s \neq \sigma(t)} \frac{\sigma(t)^2 - s^2}{\sigma(t) - s} = \lim_{s \to t,\, s \neq \sigma(t)} (\sigma(t) + s) = \sigma(t) + t = 2t + \mu^*(t)$$

is a polynomial with degree $2 - 1 = 1$. So $\mu^*$ is linear in $t$.

Equivalence (B) $\Longleftrightarrow$ (C) can be seen from a thorough and simple discussion of all the cases.

All the other implications are simple consequences of the definitions (1), (2), (3). This completes the proof.

We will call a time scale on this list a *linear graininess time scale* or *LG time scale* for short. We will further say that $\mathbb{T} = \mathbb{R}$ is *continuous*, whereas the other types are called *discrete*.

From the viewpoint of time scales Theorem 1 explains why in the theory of OPs only the three essentially different types of time scales and corresponding "diff" operators appear: the real line with continuous differentiation, equidistant grids with $h$-difference operators and $q$-grids with $q$-difference operators.

We remark that on an arbitrary LG time scale there are the following relations:

$$\sigma \circ \varrho = \mathrm{id}_{\mathbb{T}}, \tag{12}$$

$$\varrho \circ \sigma = \mathrm{id}_{\mathbb{T}}, \tag{13}$$

$$\sigma(t) - \sigma(s) = q \cdot (t - s), \qquad s, t \in \mathbb{T}. \tag{14}$$

For $\mathbb{T} = \mathbb{R}$ we just have $f^\Delta(t) = f'(t)$, whereas for the discrete time scales on the above list we have

$$f^\Delta(t) = \frac{f(qt+h) - f(t)}{(q-1)t + h}.$$

For $q = 1$ this reduces to the usual forward difference operator, whereas for $h = 0$ this is the (*Hahn* or *Jackson*) $q$-difference operator.

On LG time scales the Delta differentiation commutes with the jump operators as follows

$$f^{\Delta}(\sigma(t)) = \frac{1}{q} \cdot (f \circ \sigma)^{\Delta}(t) \tag{15}$$

$$f^{\Delta}(\varrho(t)) = q \cdot (f \circ \varrho)^{\Delta}(t). \tag{16}$$

For the reals the time scale integral just coincides with the Riemann integral

$$\int_r^s f(t)\Delta t = \int_r^s f(t)\, dt.$$

On an arbitrary discrete LG time scale we have for $\alpha, \beta \in \mathbb{Z}$, $\alpha < \beta$,

$$\begin{aligned}\int_{\sigma^{\alpha}(t_0)}^{\sigma^{\beta}(t_0)} f(t)\Delta t &= \sum_{j=\alpha}^{\beta-1} [(q-1)\sigma^j(t_0) + h] \cdot f(\sigma^j(t_0)) \\ &= \left[ -\mu^*(t) \cdot \sum_{j=0}^{\infty} q^j f(t) \right]_{t=\sigma^{\alpha}(t_0)}^{t=\sigma^{\beta}(t_0)}.\end{aligned}$$

The right expression is only valid, if the limits of the series exist. This can be checked by Delta differentiation with respect to $s$ and observing the special case of (14)

$$\mu^*(\sigma(t)) = q \cdot \mu^*(t).$$

For the convenience of the reader we write down the integral expressions for various special types of discrete LG time scales

Type II $$\int_{h\alpha+t_0}^{h\beta+t_0} f(t)\Delta t = \sum_{j=\alpha}^{\beta-1} h \cdot f(hj + t_0),$$

Type II' $$\int_{h\alpha}^{h\beta} f(t)\Delta t = \sum_{j=\alpha}^{\beta-1} h \cdot f(hj),$$

Type III' $$\int_{q^{\alpha}}^{q^{\beta}} f(t)\Delta t = \sum_{j=\alpha}^{\beta-1} (q-1)q^j \cdot f(q^j),$$

Type IV' $$\int_{-q^{\alpha}}^{-q^{\beta}} f(t)\Delta t = \sum_{j=\alpha}^{\beta-1} (1-q)q^j \cdot f(-q^j).$$

Another formula that is worth noting here is the integral shift formula

$$\int_r^s f(\sigma(t))\,\Delta t = \frac{1}{q}\cdot\int_{\sigma(r)}^{\sigma(s)} f(t)\,\Delta t \;=\; \frac{1}{q}\cdot\int_r^s f(t)\,\Delta t + \frac{\mu^*(s)f(s)-\mu^*(r)f(r)}{q}$$

that we only have to prove for the discrete case. With the abbreviation $\sigma^j := \sigma^j(t_0)$ we have

$$\begin{aligned}
&\int_{\sigma^\alpha}^{\sigma^\beta} f(\sigma(t))\,\Delta t\\
&\quad = \sum_{j=\alpha}^{\beta-1}\big((q-1)\sigma^j+h\big)\,f(\sigma^{j+1}) = \sum_{j=\alpha}^{\beta-1}\left((q-1)\frac{\sigma^{j+1}-h}{q}+h\right)f(\sigma^{j+1})\\
&\quad = \sum_{j=\alpha+1}^{\beta}\left((q-1)\frac{\sigma^j-h}{q}+h\right)f(\sigma^j) = \frac{1}{q}\cdot\sum_{j=\alpha+1}^{\beta}\big((q-1)\sigma^j+h\big)\,f(\sigma^j)\\
&\quad = \frac{1}{q}\cdot\int_{\sigma^{\alpha+1}}^{\sigma^{\beta+1}} f(t)\,\Delta t = \frac{1}{q}\cdot\int_{\sigma^\alpha}^{\sigma^\beta} f(t)\,\Delta t \;+\; \frac{\mu^*(\sigma^\beta)f(\sigma^\beta)-\mu^*(\sigma^\alpha)f(\sigma^\alpha)}{q}.
\end{aligned}$$

An alternative proof is by differentiating with respect to $s$ and using (15).

The so-called *Nabla operator* $\nabla$ is defined on an arbitrary time scale just by replacing $\sigma$ by $\varrho$ in the definition (4) or (5):

$$f^\nabla(t) := \lim_{s\to t,\, s\neq\varrho(t)} \frac{f(\varrho(t))-f(s)}{\varrho(t)-s}.$$

It is easy to check that for an arbitrary LG time scale we have

$$f^\nabla(t) := f^\Delta(\varrho(t)) = q\cdot(f\circ\varrho)^\Delta(t).$$

The time reversal operator

$$R:\begin{cases}\mathbb{T}_1=\mathbb{T}_{q,h,t_0}\to\mathbb{T}_2=\mathbb{T}_{\frac{1}{q},\frac{h}{q},-t_0}\\ \qquad t\mapsto -t\end{cases}$$

is a bijection between two LG time scales. If we denote the corresponding operator for functions by $\mathcal{R}$, i.e., $\mathcal{R}f(t) := f(-t)$, then we have the following intertwining relations:

$$\begin{aligned}R\,\sigma_1 &= \varrho_2\,R,\\ \nabla_2\,\mathcal{R} &= -\mathcal{R}\,\Delta_1,\end{aligned}$$

where $\varrho_2$ and $\nabla_2$ on $\mathbb{T}_{\frac{1}{q},\frac{h}{q},-t_0}$ are given by

$$\varrho_2(t) = \frac{t-\frac{h}{q}}{\frac{1}{q}} = qt-h,$$
$$\nabla_2 f(t) = \frac{f(t)-f(qt-h)}{(1-q)t+h}.$$

Clearly,

$$R(\sigma_1(t)) = R(qt+h) = -(qt+h) = q(-t)-h = \varrho_2(Rt),$$
$$\nabla_2\,\mathcal{R}\,f(t) = \frac{f(-t)-f(-(qt-h))}{(1-q)t+h} = -\frac{f(q(-t)+h)-f(-t)}{(q-1)(-t)+h} = -\mathcal{R}\,\Delta_1 f(t).$$

These relations would allow us to convey the calculus also to time scales $q^{\mathbb{Z}}$, $q<1$ or $-q^{\mathbb{Z}}$, $q>1$ that do not appear on the list in Theorem 1. Since this would only mean additional formal amount without further insight, we do not pursue this direction here.

More details on Delta and Nabla calculus can be found in [10–14].

## 4. OPs on LG Time Scales

In this section we define OPs on LG time scales and list a few main properties.

From now on let $J \subseteq \mathbb{T}$ be an infinite interval in an LG time scale, that does not have a left-scattered maximum. This condition makes sure that the derivative of a differentiable function on $J$ will have $J$ as its domain as well. Skipping this condition is possible, but will result in additional technical and notational efforts (bookkeeping of domains, kappa operator), which are not justified by the mathematical outcome.

Let us define $r := \inf\ J$, $s := \sup J$. For a function $f : J \to \mathbb{R}$ we will use the common abbreviation

$$f(r) = \lim_{t\searrow r} f(t), \qquad f(s) = \lim_{t\nearrow s} f(t),$$

tacitly assuming then that the limits exist. Now, consider the sequence of orthonormal polynomials $p_n : J \to \mathbb{R}$, where $n \in \mathbb{N}_0$ is the degree. They satisfy the orthonormality condition

$$\int_r^s p_n(t)p_m(t)w(t)\,\Delta t = \delta_{n,m}, \tag{17}$$

where $\delta_{n,m}$ is Kronecker's delta and $w : J \to \mathbb{R}$ is a certain differentiable weight function, which is positive on $]r,s[$. We assume that all moments $\int_r^s t^j w(t)\Delta t$, $j \in \mathbb{N}_0$, are finite. Clearly, in case $J = \mathbb{R}$, we have the usual definition of orthonormal polynomials on the real line. If $J$ is discrete we get $h$- or $q$-type orthonormal polynomials.

Applying the inner product (17) to the ansatz

$$tp_n(t) = \sum_{j=0}^{n+1} c_{nj} p_j(t)$$

yields the three-term recurrence relation

$$tp_n(t) = a_{n+1}\, p_{n+1}(t) + b_n\, p_n(t) + a_n\, p_{n-1}(t) \tag{18}$$

with

$$a_0 = 0, \quad a_n = \int_r^s tp_n(t)p_{n-1}(t)w(t)\Delta t, \quad b_n = \int_r^s tp_n^2(t)w(t)\Delta t.$$

Comparing the highest coefficients in (18) yields the identity

$$a_{n+1} \cdot \gamma_{n+1} = \gamma_n, \tag{19}$$

where $\gamma_n$ is the leading coefficient in $p_n$. Note that in [8] the sequence of polynomials defined on $\mathbb{T}$ and satisfying (18) was called an OP system on the time scale $\mathbb{T}$.

Another useful identity is the Christoffel–Darboux formula, which we next prove to be self-contained. It is a statement about the polynomial with degree $n$ in each variable

$$\mathcal{P}_n(x, y) := \sum_{j=0}^{n} p_j(x)p_j(y) = a_{n+1} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x - y}. \tag{20}$$

The formula is easily proved by induction. By comparing the coefficients in (18) we readily get the identity in case $n = 0$. The induction step is proved as follows. From (18) we get

$$\begin{aligned} xp_n(x)p_n(y) &= a_{n+1}p_{n+1}(x)p_n(y) + b_n p_n(x)p_n(y) + a_n p_{n-1}(x)p_n(y), \\ yp_n(y)p_n(x) &= a_{n+1}p_{n+1}(y)p_n(x) + b_n p_n(y)p_n(x) + a_n p_{n-1}(y)p_n(x). \end{aligned}$$

Subtracting these equalities and applying the induction hypothesis to the terms with the coefficient $a_n$, we get (20).

## 5. Ladder Operators

The raising and lowering operators for classical and semi-classical OPs can simply be defined as the ones which raise or lower the degree of the OP. See [2–5] for examples. They can be found for multiple OPs [18] orthogonal with respect to several measures and matrix OPs [19,20]. Some spectral theoretical properties of ladder theory for OPs systems on time scales are considered in [8].

**Theorem 2.** *The polynomials $\{p_n(x)\}$ satisfy*

$$p_n^{\Delta}(x) = A_n(x)p_{n-1}(x) - B_n(x)p_n(x), \quad x \in J, \tag{21}$$

*where the functions $A_n, B_n : J \to \mathbb{R}$ are given by*

$$\begin{aligned} A_n(x) &= \frac{-a_n p_n(y)p_n(\varrho(y))w(\varrho(y))}{x-\varrho(y)}\Big|_{y=r}^{y=s} \\ &\quad + a_n \int_r^s p_n(t)p_n(\varrho(t))\frac{u(t)-u(\sigma(x))}{t-\sigma(x)}w(t)\Delta t, \\ B_n(x) &= \frac{-a_n p_n(y)p_{n-1}(\varrho(y))w(\varrho(y))}{x-\varrho(y)}\Big|_{y=r}^{y=s} \\ &\quad + a_n \int_r^s p_n(t)p_{n-1}(\varrho(t))\frac{u(t)-u(\sigma(x))}{t-\sigma(x)}w(t)\Delta t \end{aligned}$$

*and $u : J \to \mathbb{R}$ is a function that satisfies*

$$u(t) \cdot w(t) \; = \; -w^{\Delta}(\varrho(t)). \tag{22}$$

This theorem gives the lowering (annihilation) operator studied later on. Note that in case of a discrete time scale the terms containing $p_n(\varrho(r))p_n(r)w(\varrho(r))$ cancel in the above expressions for $A_n$ and $B_n$, so their values are irrelevant. Since $s = +\infty$ or $s$ is left-dense, we have $\varrho(s) = s$ in any case.

It is not difficult to thoroughly check that in the different cases Theorem 2 precisely coincides with the results in the following references:

[2] Thm. 2.1, for $\mathbb{T} = \mathbb{R}$ (Type I).
[3] Thm. 1.1, equations (1.10), (1.11), for $\mathbb{T} = \mathbb{Z}$ (Type II').
[5] Thm 2.1, for $\mathbb{T} = h\mathbb{Z} + t_0$ (Type II). Note that in [5] the gap length = graininess $h$ is only relevant for the underlying grid (= time scale), but does not enter the definitions of the difference operator and the integral. Hence already the normalization (2.1) there differs from our definition (17) by a factor $h$.
[4] Thm 2.1, for $\mathbb{T} = q^{\mathbb{Z}}$ (Types III', IV'). The authors do not specify an underlying time scale explicitly, they only make implicit use of the condition $q < 1$ in their definition of the integral (1.7) by infinite sums as in (17). Formula (2.4) in [4] is valid for all $x \in \mathbb{R}$, provided the involved infinite sums exist. The corresponding formula (21) here is valid only for $x \in \mathbb{T}$, but we don't have to take care about convergence of integrals with finite bounds. If the function $u$ defined in (22) is a polynomial, then (21) can be extended to all of $\mathbb{R}$ anyway. We remark that in [4] there are sign misprints in the boundary terms of the definitions (2.5), (2.6) of $A_n(x)$ and $B_n(x)$.

*Proof.* As we mentioned above the proof is modelled on the proofs in [2–4]. Let

$$p_n^{\Delta}(x) = \sum_{k=0}^{n-1} c_{nk} p_k(x). \tag{23}$$

The orthonormality relation (17), the integration by parts formulas (11), the product rule (7) and formulas (16), (12)–(14) give

$$\begin{aligned}
c_{nk} &= \int_r^s p_n^{\Delta}(t) p_k(t) w(t) \Delta t \\
&= \int_r^s p_n^{\Delta}(t) p_k(\varrho(\sigma(t))) w(\varrho(\sigma(t))) \Delta t \\
&= p_n(y) p_k(\varrho(y)) w(\varrho(y)) \Big|_{y=r}^{y=s} - \int_r^s p_n(t) [(p_k \cdot w) \circ \varrho]^{\Delta}(t) \Delta t \\
&= p_n(y) p_k(\varrho(y)) w(\varrho(y)) \Big|_{y=r}^{y=s} - \frac{1}{q} \cdot \int_r^s p_n(t) [p_k \cdot w]^{\Delta}(\varrho(t)) \Delta t \\
&= p_n(y) p_k(\varrho(y)) w(\varrho(y)) \Big|_{y=r}^{y=s} - \frac{1}{q} \cdot \int_r^s p_n(t) [p_k^{\Delta}(\varrho(t)) w(\sigma(\varrho(t))) \\
&\quad + p_k(\varrho(t)) w^{\Delta}(\varrho(t))] \Delta t \\
&= p_n(y) p_k(\varrho(y)) w(\varrho(y)) \Big|_{y=r}^{y=s} - \frac{1}{q} \cdot \int_r^s p_n(t) \Big( p_k^{\Delta}(\varrho(t)) - p_k(\varrho(t)) u(t) \Big) w(t) \Delta t \\
&= p_n(y) p_k(\varrho(y)) w(\varrho(y)) \Big|_{y=r}^{y=s} + \frac{1}{q} \cdot \int_r^s p_n(t) p_k(\varrho(t)) [u(t) - u(\sigma(x))] w(t) \Delta t.
\end{aligned}$$

We once more used the orthogonality relation in the last step, since $p_k^{\Delta}(\varrho(t))$ and $p_k(\varrho(t))$ are polynomials of degree less than $n-1$. Next plugging the coefficients into (23), using the Christoffel–Darboux formula (20) and formula (14) again, we get

$$\begin{aligned}
p_n^{\Delta}(x) &= \sum_{k=0}^{n-1} c_{nk} p_k(x) \\
&= \sum_{k=0}^{n-1} p_k(x) p_k(\varrho(y)) p_n(y) w(\varrho(y)) \Big|_{y=r}^{y=s} \\
&\quad + \frac{1}{q} \cdot \int_r^s p_n(t) \left( \sum_{k=0}^{n-1} p_k(x) p_k(\varrho(t)) \right) [u(t) - u(\sigma(x))] w(t) \Delta t
\end{aligned}$$

$$
\begin{aligned}
&= a_n\, p_n(y)\, \frac{p_n(x)p_{n-1}(\varrho(y)) - p_{n-1}(x)p_n(\varrho(y))}{x-\varrho(y)} w(\varrho(y))\Big|_{y=r}^{y=s} \\
&\quad + \frac{a_n}{q}\int_r^s p_n(t)[p_n(\varrho(t))p_{n-1}(x) - p_{n-1}(\varrho(t))p_n(x)]\frac{u(t)-u(\sigma(x))}{\varrho(t)-x} w(t)\Delta t \\
&= a_n \frac{p_n(x)p_{n-1}(\varrho(y)) - p_n(\varrho(y))p_{n-1}(x)}{x-\varrho(y)}\, p_n(y)w(\varrho(y))\Big|_{y=r}^{y=s} \\
&\quad + a_n\int_r^s p_n(t)[p_n(\varrho(t))p_{n-1}(x) - p_{n-1}(\varrho(t))p_n(x)]\frac{u(t)-u(\sigma(x))}{t-\sigma(x)} w(t)\Delta t \\
&= \left[\frac{-a_n p_n(y)p_n(\varrho(y))w(\varrho(y))}{x-\varrho(y)}\Big|_{y=r}^{y=s}\right. \\
&\quad \left. + a_n \int_r^s p_n(t)p_n(\varrho(t))\frac{u(t)-u(\sigma(x))}{t-\sigma(x)} w(t)\Delta t\right]\cdot p_{n-1}(x) \\
&\quad - \left[\frac{-a_n p_n(y)p_{n-1}(\varrho(y))w(\varrho(y))}{x-\varrho(y)}\Big|_{y=r}^{y=s}\right. \\
&\quad \left. + a_n \int_r^s p_n(t)p_{n-1}(\varrho(t))\frac{u(t)-u(\sigma(x))}{t-\sigma(x)} w(t)\Delta t\right]\cdot p_n(x) \\
&= A_n(x)\cdot p_{n-1}(x) - B_n(x)\cdot p_n(x).
\end{aligned}
$$

This completes the proof.

Inserting the three-term recurrence (18) into equation (21) from Theorem 2 yields

$$
\begin{aligned}
p_n^{\Delta}(x) + B_n(x)p_n(x) &= A_n(x)p_{n-1}(x) \\
&= \frac{A_n(x)}{a_n}\Big((x-b_n)p_n(x) - a_{n+1}p_{n+1}(x)\Big), \quad x \in J.
\end{aligned}
$$

This motivates the definition of two (lowering and raising) first order "diff" operators for differentiable functions $J \to \mathbb{R}$

$$
\begin{aligned}
L_n^- g(x) &:= g^{\Delta}(x) + B_n(x)g(x), \\
L_n^+ g(x) &:= -g^{\Delta}(x) - B_n(x)g(x) + (x-b_n)\frac{A_n(x)}{a_n}g(x).
\end{aligned}
$$

Then the above equations read as

$$
\begin{aligned}
L_n^- p_n(x) &= A_n(x)p_{n-1}(x), \\
L_{n+1}^+ p_n(x) &= a_{n+1}\frac{A_n(x)}{a_n}p_{n+1}(x).
\end{aligned}
$$

When $A_n(x) \neq 0$ we combine these operators in order to get two second order dynamic equations

$$L_n^+\Big(\frac{a_n}{A_n(x)}L_n^-\Big)p_n(x) = a_n^2\frac{A_{n-1}(x)}{a_{n-1}}p_n(x),$$
$$L_{n+1}^-\Big(\frac{a_n}{A_n(x)}L_{n+1}^+\Big)p_n(x) = a_{n+1}^2\frac{A_{n+1}(x)}{a_{n+1}}p_n(x)$$

for the OPs in a similar way as in [3]. These equations can be further elaborated by using the product rule (7) and quotient rule (8) in connection with the "simple useful formula" (6). See also [3,4] for this direction. We also note that we can study the Nabla operator in a similar way for OPs on LG time scales and obtain the analogous formulas for the ladder operators. See [7] for this direction. Another more algebraic approach to a ladder formalism for OPs can be found in [21].

## 6. Recurrence Relations

Next we derive discrete equations satisfied by $A_n(x)$ and $B_n(x)$ (with respect to the index $n$). For simplicity in the following we assume that the weight function vanishes at the boundary points, more precisely this means $w(\varrho(r)) = 0$ and $w(\varrho(s)) = w(s) = 0$. The proof with the boundary terms can easily be repeated and we omit the results. Our proof is similar to the proofs in [2,6].

**Theorem 3.** *The coefficients $A_n(x)$ and $B_n(x)$ in the lowering operator and the coefficients in the three-term recurrence relation (18) satisfy the following relations:*

$$B_{n+1}(x) + B_n(x) = (x - b_n)\frac{A_n(x)}{a_n} + \mu^*(x)\sum_{j=0}^{n}\frac{A_j(x)}{a_j} - u(\sigma(x)), \tag{24}$$

$$(x-b_n)B_{n+1}(x)-(\sigma(x)-b_n)B_n(x)=n(q-1)-1+a_{n+1}^2\frac{A_{n+1}(x)}{a_{n+1}}-a_n^2\frac{A_{n-1}(x)}{a_{n-1}}. \tag{25}$$

*Proof.* In order to prepare the proofs we first recall the expressions from Theorem 1 without boundary terms:

$$A_n(x) = a_n\int_r^s p_n(\varrho(t))p_n(t)\frac{u(\sigma(x)) - u(t)}{\sigma(x) - t}w(t)\,\Delta t,$$
$$B_n(x) = a_n\int_r^s p_{n-1}(\varrho(t))p_n(t)\frac{u(\sigma(x)) - u(t)}{\sigma(x) - t}w(t)\,\Delta t.$$

We will repeatedly make use of (18) in the following expression:

$$(t - b_n)p_n(t) = a_np_{n-1}(t) + a_{n+1}p_{n+1}(t). \tag{26}$$

In addition to (17) there are the following orthogonality relations for $k \le n+1$:

$$\int_r^s p_k(t)\, p_n(\varrho(t))\, w(t)\, \Delta t = \frac{1}{q^n} \cdot \delta_{k,n}, \tag{27}$$

$$\int_r^s p_k(t)\, p_n(\varrho(t))\, u(t)\, w(t)\, \Delta t = \frac{q\,(n+1)}{a_{n+1}} \cdot \delta_{k,n+1}. \tag{28}$$

The first relation (27) follows from

$$p_n(\varrho(t)) = \frac{1}{q^n}\, p_n(t) + \text{lower order terms}.$$

By using (22), (11), then (12)–(14) and (7) and then the orthonormality condition (17), we can prove the other relation (28) as follows:

$$\begin{aligned}
\int_r^s p_k(t)\, p_n(\varrho(t))\, u(t)\, w(t)\, \Delta t &= -\int_r^s p_k(t)\, p_n(\varrho(t))\, w^{\Delta}(\varrho(t)\, \Delta t \\
&= -q \int_r^s p_k(t)\, p_n(\varrho(t))\, (w \circ \varrho)^{\Delta}(t)\, \Delta t \\
&= q \int_r^s [p_k \cdot (p_n \circ \varrho)]^{\Delta}(t)\, (w \circ \varrho)(\sigma(t))\, \Delta t \\
&= q \int_r^s [p_k(t) p_n^{\Delta}(\varrho(t)) + p_k^{\Delta}(t) p_n(t)]\, w(t)\, \Delta t \\
&= q \int_r^s p_k^{\Delta}(t)\, p_n(t)\, w(t)\, \Delta t \\
&= \frac{q\,(n+1)}{a_{n+1}} \cdot \delta_{k,n+1},
\end{aligned}$$

since by (19)

$$\begin{aligned}
p_{n+1}^{\Delta}(t) &= (n+1) \frac{\gamma_{n+1}}{\gamma_n}\, p_n(t) + \text{lower order terms} \\
&= \frac{n+1}{a_{n+1}}\, p_n(t) + \text{lower order terms}.
\end{aligned}$$

Also observe the time scale identities which will be used later on:

$$\begin{aligned}
t-\varrho(t) &= \mu^*(x)-(\sigma(x)-x)-(\varrho(t)-t),\\
&= \mu^*(x)+\frac{\sigma(x)-t}{q}-(\sigma(x)-t) \;=\; \mu^*(x)+\frac{1-q}{q}(\sigma(x)-t),\\
\varrho(t)-b_n &= x-b_n-\frac{\sigma(x)-t}{q},\\
\sigma(x)-b_n &= (\sigma(x)-t)+(t-b_n).
\end{aligned}$$

Now with the orthogonality relations (17), (27), (28) and the Christoffel–Darboux formula (20) we have

$$\begin{aligned}
B_n(x)+B_{n+1}(x) &= a_n\int_r^s p_{n-1}(\varrho(t))p_n(t)\frac{u(\sigma(x))-u(t)}{\sigma(x)-t}w(t)\Delta t\\
&\quad + a_{n+1}\int_r^s p_n(\varrho(t))p_{n+1}(t)\frac{u(\sigma(x))-u(t)}{\sigma(x)-t}w(t)\Delta t\\
&= \int_r^s (\varrho(t)-b_n)p_n(\varrho(t))p_n(t)\frac{u(\sigma(x))-u(t)}{\sigma(x)-t}w(t)\Delta t\\
&\quad + a_{n+1}\int_r^s [p_n(\varrho(t))p_{n+1}(t)\\
&\quad -p_{n+1}(\varrho(t))p_n(t)]\frac{u(\sigma(x))-u(t)}{\sigma(x)-t}w(t)\Delta t\\
&= -\int_r^s \frac{\sigma(x)-t}{q}\,p_n(\varrho(t))p_n(t)\frac{u(\sigma(x))-u(t)}{\sigma(x)-t}w(t)\Delta t\\
&\quad + (x-b_n)\int_r^s p_n(\varrho(t))p_n(t)\frac{u(\sigma(x))-u(t)}{\sigma(x)-t}w(t)\Delta t\\
&\quad + \int_r^s (t-\varrho(t))\sum_{j=0}^n p_j(t)p_j(\varrho(t))\frac{u(\sigma(x))-u(t)}{\sigma(x)-t}w(t)\Delta t\\
&= -\frac{u(\sigma(x))}{q^{n+1}}+(x-b_n)\,\frac{A_n(x)}{a_n}\\
&\quad + \mu^*(x)\int_r^s \sum_{j=0}^n p_j(t)p_j(\varrho(t))\frac{u(\sigma(x))-u(t)}{\sigma(x)-t}w(t)\Delta t
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1-q}{q}\int_r^s\sum_{j=0}^n p_j(t)p_j(\varrho(t))u(\sigma(x))w(t)\Delta t\\
&-\frac{1-q}{q}\int_r^s\sum_{j=0}^n p_j(t)p_j(\varrho(t))u(t)w(t)\Delta t\\
=&-\frac{u(\sigma(x))}{q^{n+1}}+(x-b_n)\,\frac{A_n(x)}{a_n}+\mu^*(x)\sum_{j=0}^n\frac{A_j(x)}{a_j}\\
&+\frac{1-q}{q}\,u(\sigma(x))\sum_{j=0}^n q^{-j}\\
=&\;(x-b_n)\,\frac{A_n(x)}{a_n}+\;\mu^*(x)\sum_{j=0}^n\frac{A_j(x)}{a_j}-u(\sigma(x)).
\end{aligned}
$$

For the proof of (25) we apply (26). Finally we use the orthogonality relation (28) and insert the definition of $A_n(x)$. We have

$$
\begin{aligned}
&(x-b_n)B_{n+1}(x)-(\sigma(x)-b_n)B_n(x)\\
&=a_{n+1}\int_r^s\frac{\sigma(x)-t}{q}p_n(\varrho(t))p_{n+1}(t)\frac{u(\sigma(x))-u(t)}{\sigma(x)-t}w(t)\,\Delta t\\
&\quad+a_{n+1}\int_r^s(\varrho(t)-b_n)p_n(\varrho(t))p_{n+1}(t)\frac{u(\sigma(x))-u(t)}{\sigma(x)-t}w(t)\,\Delta t\\
&\quad-a_n\int_r^s(\sigma(x)-t)p_{n-1}(\varrho(t))p_n(t)\frac{u(\sigma(x))-u(t)}{\sigma(x)-t}w(t)\,\Delta t\\
&\quad-a_n\int_r^s(t-b_n)p_{n-1}(\varrho(t))p_n(t)\frac{u(\sigma(x))-u(t)}{\sigma(x)-t}w(t)\,\Delta t\\
&=-(n+1)+\;a_{n+1}\int_r^s[a_np_{n-1}(\varrho(t))\\
&\quad+a_{n+1}p_{n+1}(\varrho(t))]\,p_{n+1}(t)\frac{u(\sigma(x))-u(t)}{\sigma(x)-t}w(t)\,\Delta t\\
&\quad+nq-\;a_n\int_r^s[a_np_{n-1}(t)+a_{n+1}p_{n+1}(t)]\,p_{n-1}(\varrho(t))\frac{u(\sigma(x))-u(t)}{\sigma(x)-t}w(t)\,\Delta t\\
&=n(q-1)-1+a_{n+1}^2\frac{A_{n+1}(x)}{a_{n+1}}-a_n^2\frac{A_{n-1}(x)}{a_{n-1}}.
\end{aligned}
$$

Again, it is straightforward to check that the results of the theorem coincide with the known relations in [2,3] for the respective time scales. This completes the proof.

## 7. Functions of the Second Kind

Finally, we define the functions $Q_n(x)$ of the second kind as the Cauchy–Hilbert transforms of $p_n$

$$Q_n(x) = \frac{1}{w(x)} \int_r^s \frac{p_n(y)}{(x-y)}\, w(y)\, \Delta y. \tag{29}$$

The functions fulfill the recurrence relations in the following form

$$\begin{aligned} xQ_0(x) &= a_1 Q_1(x) + b_0 Q_0(x) + \frac{1}{\gamma_0 w(x)}, \\ xQ_n(x) &= a_{n+1} Q_{n+1}(x) + b_n Q_n(x) + a_n Q_{n-1}(x), \quad n \geq 1. \end{aligned}$$

The first relation follows from the identity

$$\int_r^s \frac{(x-b_0)-(y-b_0)}{x-y} \gamma_0^2 w(y) \Delta\, y \ = \ 1,$$

which is equivalent to

$$\frac{x-b_0}{w(x)} \int_r^s \frac{\gamma_0}{x-y} w(y) \Delta\, y \ = \ \frac{a_1}{w(x)} \int_r^s \frac{y-b_0}{a_1(x-y)} \gamma_0 w(y) \Delta\, y + \frac{1}{\gamma_0 w(x)}$$

and then to

$$\frac{x-b_0}{w(x)} \int_r^s \frac{w(y)}{x-y} p_0(y) \Delta\, y \ = \ \frac{a_1}{w(x)} \int_r^s \frac{w(y)}{x-y} p_1(y) \Delta\, y + \frac{1}{\gamma_0 w(x)}$$

using the three-term recurrence relation. Recall also that $p_0(y)$ is just a constant and $a_1 = \gamma_0/\gamma_1$, where $\gamma_n$ is the leading coefficient of $p_n(x)$. Now observe that due to orthogonality we have

$$\begin{aligned} &\int_r^s y \frac{w(y)}{(x-y)w(x)}\, p_n(y)\, \Delta y \\ &= \int_r^s y \frac{w(y)}{(x-y)w(x)}\, p_n(y)\, \Delta y + \int_r^s (x-y) \frac{w(y)}{(x-y)w(x)}\, p_n(y)\, \Delta y \\ &= x \int_r^s \frac{w(y)}{(x-y)w(x)}\, p_n(y)\, \Delta y \end{aligned}$$

and so the three term recurrence relation for $Q_n$ $(n \geq 1)$ is just the Cauchy Hilbert-transform of the original three term recurrence relation (18).

**Theorem 4.** *The function* (29) *satisfies the lowering dynamic equation* (21).

*Proof.* We will now prove, as in [7, Th. 3.1], that the lowering relation (21) for the functions $p_n(x)$, i.e., equation

$$p_n^\Delta(x) = A_n(x)p_{n-1}(x) - B_n(x)p_n(x) \tag{30}$$

is preserved under the Cauchy–Hilbert transform, that is,

$$Q_n^\Delta(x) = A_n(x)Q_{n-1}(x) - B_n(x)Q_n(x). \tag{31}$$

Note that this automatically implies that these functions satisfy the raising equation as well.

The idea of the proof is as follows. We apply the (modified) Cauchy–Hilbert transform (multiplied by $w(x)$ to shorten the formulas) to both sides of (30). We then add the correction term

$$\int_r^s [u(\sigma(x) - u(y)] \frac{w(y)}{\sigma(x) - y} p_n(y)\Delta y \tag{32}$$

to both sides in order to arrive at

$$w(x)Q_n^\Delta(x) = A_n(x)w(x)Q_{n-1}(x) - B_n(x)w(x)Q_n(x).$$

First we compute the (modified) Cauchy–Hilbert transform of $p_n^\Delta$ by the partial integration, the product and quotient rule:

$$\begin{aligned}
\int_r^s \frac{w(y)}{x-y} p_n^\Delta(y)\Delta y &= \int_r^s \frac{w(\rho(\sigma(y)))}{x - \rho(\sigma(y))} p_n^\Delta(y)\Delta y \\
&= -\int_r^s \left(\frac{w(\rho(y))}{x-\rho(y)}\right)^{\Delta_y} p_n(y)\Delta y \\
&= -\frac{1}{q}\int_r^s \left(\frac{w(z)}{x-z}\right)^{\Delta_z} \Big|_{z=\rho(y)} p_n(y)\Delta y \\
&= -\frac{1}{q}\int_r^s \frac{w^\Delta(z)(x-z) + w(z)}{(x-z)(x-\sigma(z))}\Big|_{z=\rho(y)} p_n(y)\Delta y \\
&= -\int_r^s \frac{w^\Delta(\rho(y))(x-\rho(y)) + w(\rho(y))}{(\sigma(x)-y)(x-y)} p_n(y)\Delta y \\
&= \int_r^s \frac{u(y)(x-y) - 1}{(\sigma(x)-y)(x-y)} w(y)p_n(y)\Delta y.
\end{aligned}$$

The last equality follows from the definition of $u$ and the formula

$$w(\rho(y)) \;=\; w(y) - \mu^*(\rho(y))w^{\Delta}(\rho(y)).$$

Now adding the correction term (32) yields

$$\begin{aligned}
&\int_r^s \frac{(x-y)u(y)-1}{(x-y)(\sigma(x)-y)} p_n(y)w(y)\,\Delta y + \int_r^s \frac{u(\sigma(x))-u(y)}{\sigma(x)-y} p_n(y)w(y)\,\Delta y \\
&= \int_r^s \left[u(\sigma(x)) - u(y) + \frac{(x-y)u(y)-1}{x-y}\right] \frac{p_n(y)w(y)}{\sigma(x)-y}\Delta y \\
&= \int_r^s \left[-\frac{w^{\Delta}(x)}{w(\sigma(x))} - \frac{1}{x-y}\right] \frac{p_n(y)w(y)}{\sigma(x)-y}\Delta y \\
&= -\frac{w^{\Delta}(x)}{w(\sigma(x))} \cdot \int_r^s \frac{p_n(y)w(y)}{\sigma(x)-y}\Delta\, y - \int_r^s \frac{p_n(y)w(y)}{(x-y)(\sigma(x)-y)}\Delta y \\
&= w(x)\left[\frac{1}{w(x)} \cdot \int_r^s \frac{p_n(y)w(y)}{x-y}\,\Delta y\right]^{\Delta} \\
&= w(x)Q_n^{\Delta}(x).
\end{aligned}$$

Using the Christoffel–Darboux polynomial $\mathcal{P}_{n-1}$ (20) and orthogonality, we compute the (modified) Cauchy–Hilbert transform of the right hand side of (30):

$$\begin{aligned}
&\int_r^s \left[A_n(y)p_{n-1}(y) - B_n(y)p_n(y)\right] \frac{w(y)\Delta y}{x-y} \\
&\quad = -a_n \int_r^s \left( \int_r^s \frac{u(\sigma(y))-u(t)}{\sigma(y)-t} [p_{n-1}(\rho(t))p_n(y) \right. \\
&\qquad \left. - p_n(\rho(t))p_{n-1}(y)]\; p_n(t)w(t)\Delta t \right) \frac{w(y)\Delta y}{x-y} \\
&\quad = -\frac{a_n}{q} \int_r^s \int_r^s [u(\sigma(y)) - u(t)] \frac{p_n(y)p_{n-1}(\rho(t)) - p_{n-1}(y)p_n(\rho(t))}{y-\rho(t)} \\
&\qquad \times p_n(t)w(t)\Delta t \frac{w(y)\Delta y}{x-y} \\
&\quad = -\frac{1}{q} \int_r^s \int_r^s [u(\sigma(y)) - u(t)]\mathcal{P}_{n-1}(y,\rho(t))\; p_n(t)w(t)\Delta t\; \frac{w(y)\Delta y}{x-y}
\end{aligned}$$

$$= -\frac{1}{q}\int_r^s\int_r^s [u(\sigma(x)) - u(t)]\mathcal{P}_{n-1}(y,\rho(t))\, p_n(t)w(t)\Delta t\, \frac{w(y)\Delta y}{x-y}. \quad (33)$$

We note that by orthogonality

$$\int_r^s \mathcal{P}_{n-1}(y,\rho(t))w(y)\Delta y = \sum_{j=0}^{n-1} p_j(\rho(t))\int_r^s p_j(y)w(y)\Delta y = \int_r^s \gamma_0^2 w(y)\Delta y = 1,$$

hence the correction term (32) can be represented in the form

$$\int_r^s \frac{u(\sigma(x)) - u(t)}{\sigma(x)-t}\, p_n(t)w(t)\Delta t$$
$$= \int_r^s\int_r^s \frac{u(\sigma(x)) - u(t)}{\sigma(x)-t}\mathcal{P}_{n-1}(y,\rho(t))\, p_n(t)w(t)\Delta t\, w(y)\Delta y.$$

Now adding this term to (33) yields

$$\int_r^s\int_r^s \frac{u(\sigma(x))-u(t)}{\sigma(x)-t}\left[(x-y)-\frac{\sigma(x)-t}{q}\right]\mathcal{P}_{n-1}(y,\rho(t))\, p_n(t)w(t)\Delta t\frac{w(y)\Delta y}{x-y}$$
$$= \int_r^s\int_r^s \frac{u(\sigma(x)) - u(t)}{\sigma(x)-t}\,[\rho(t)-y]\,\mathcal{P}_{n-1}(y,\rho(t))\, p_n(t)w(t)\Delta t\frac{w(y)\Delta y}{x-y}$$
$$= \int_r^s\left(\int_r^s \frac{u(\sigma(x)) - u(t)}{\sigma(x)-t}\, a_n\Big(p_n(\rho(t))p_{n-1}(y)\right.$$
$$\left. - p_{n-1}(\rho(t))p_n(y)\Big)\, p_n(t)w(t)\Delta t\right)\frac{w(y)\Delta y}{x-y}$$
$$= A_n(x)w(x)Q_{n-1}(x) - B_n(x)w(x)Q_n(x).$$

This completes the proof.

Finally we note that the results of this section generalize the results in [3, Th. 4.1] for $\mathbb{T} = \mathbb{Z}$, [4, Th. 6.1] for $\mathbb{T} = q^{\mathbb{Z}}$ and [5, Th. 2.4] for $\mathbb{T} = h\mathbb{Z} + t_0$.

## Discussions

The results of this paper will be useful for the theory of OPs since they show how one can unify and generalize seemingly different proofs for different types of OPs by using the theory of time scales calculus. In addition, the study of linear graininess provides a new avenue of research in the theory of time scales calculus. As discussed in the paper, one can generalize the results to Nabla calculus and derive more dynamic equations for OPs on the time scales. It might be interesting to further study exponential functions (in the time scales calculus) for the classical weights of OPs. The Cauchy transform on the time scales and the second kind functions associated to OPs deserve further studies as well.

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